Math 202(01) — Solution
Practice Exam I — 16 February 2004

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1. Given $G(x) = \int_3^{2x-1} e^{t^2}\,dt$, find $G'(2)$.

   Solution: Let $H(u) = \int_3^{u} e^{t^2}\,dt$. Then by the Fundamental Theorem of Calculus (Part I), we know that

   $$H'(u) = e^{u^2}$$

   Since $G(x) = H(2x-1)$, we can use the chain rule to get

   $$\begin{aligned} G'(x) &= \frac{d}{dx}(H(2x-1)) \\ &= H'(2x-1) \times 2 \\ &= e^{(2x-1)^2} \times 2 \end{aligned}$$

   Thus

   $$G'(2) = 2e^9$$

2. Find $\int_1^4 \frac{t^2+5}{2t}\,dt$

   Solution: We have

   $$\begin{aligned} \int_1^4 \frac{t^2+5}{2t}\,dt &= \int_1^4 \frac{t^2}{2t} + \frac{5}{2t}\,dt \\ &= \int_1^4 \frac{1}{2}t + \frac{5}{2}\cdot\frac{1}{t}\,dt \\ &= \left.\frac{t^2}{4} + \frac{5}{2}\ln t\right]_1^4 \\ &= 4 + \frac{5}{2}\ln 4 - \left(\frac{1}{4} + 0\right) \\ &= \frac{15}{4} + \frac{5\ln 4}{2} \end{aligned}$$

3. Find $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$.

   Solution: Let $u = \sqrt{x}$. Then

$$\begin{aligned} du &= \frac{1}{2}x^{-\frac{1}{2}}\,dx \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

   so that $2du = \frac{1}{\sqrt{x}}\,dx$. The integral becomes

$$2\int \cos(u)\,du \quad = \quad 2\sin(u) + C$$

   Undoing the substitution, we get the final answer: $2\sin(\sqrt{x}) + C$.

4. Find $\int_3^4 x^3(x^2-8)^{\frac{2}{3}}\,dx$

   Solution: Let $u = x^2 - 8$. Then $du = 2x\,dx$, so that $x\,dx = \frac{1}{2}\,du$. We'll also want to know that $x^2 = u + 8$.

   When $x = 3$, we get $u = 1$, and when $x = 4$, we get $u = 8$.

   We can rewrite the integral as

$$\begin{aligned} \int_1^8 x^2(x^2-8)^{\frac{2}{3}} x\,dx &= \frac{1}{2}\int_1^8 (u+8)u^{\frac{2}{3}}\,du \\ &= \frac{1}{2}\int_1^8 u^{\frac{5}{3}} + 8u^{\frac{2}{3}}\,du \\ &= \frac{1}{2}\left[\frac{3u^{\frac{8}{3}}}{8} + \frac{24u^{\frac{5}{3}}}{5}\right]_1^8 \\ &= \frac{1}{2}\left[\left(\frac{3\times 2^8}{8} + \frac{24\times 2^5}{5}\right) - \left(\frac{3}{8} + \frac{24}{5}\right)\right] \\ &= \frac{9777}{40} \end{aligned}$$

5. Find the area bounded by the curves $y = x^2 - x$ and $y = 2x - 2$.

Solution: At right is a sketch of the two curves and the region they bound. To find the intersection points, we solve

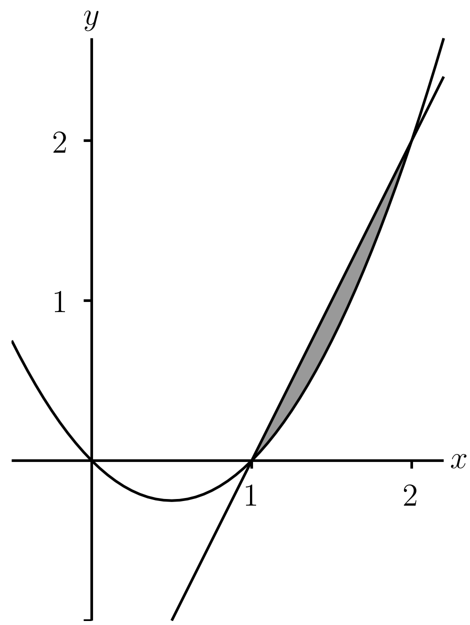
$$x^2 - x = 2x - 2$$

Moving all the terms to one side gives

$$\begin{aligned} 0 &= x^2 - 3x + 2 \\ &= (x-2)(x-1) \end{aligned}$$

so the curves intersect at $x = 1$ and $x = 2$. The line is on top throughout the interval, so the area is

$$\int_1^2 (2x-2) - (x^2 - x)\, dx$$

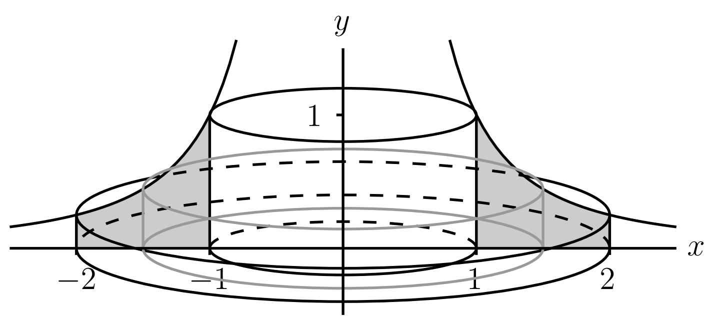


The integral simplifies to

$$\begin{aligned} \int_1^2 -x^2 + 3x - 2\, dx &= -\frac{x^3}{3} + \frac{3x^2}{2} - 2x\Big]_1^2 \\ &= -\frac{8}{3} + 6 - 4 - \left(-\frac{1}{3} + \frac{3}{2} - 2\right) \\ &= \frac{1}{6} \end{aligned}$$

6. Let R be the region bounded by the curve $y = \dfrac{1}{x^2}$ and the lines $x = 1$, $x = 2$, and $y = 0$. Set up, but do not evaluate, an integral for the volume of the solid generated when the region R is revolved about the y-axis.

Solution: Here is a picture of the region. The integral will be simplest if we use cylindrical shells. The shell at $x = \dfrac{1}{2}$ is shown in the picture.

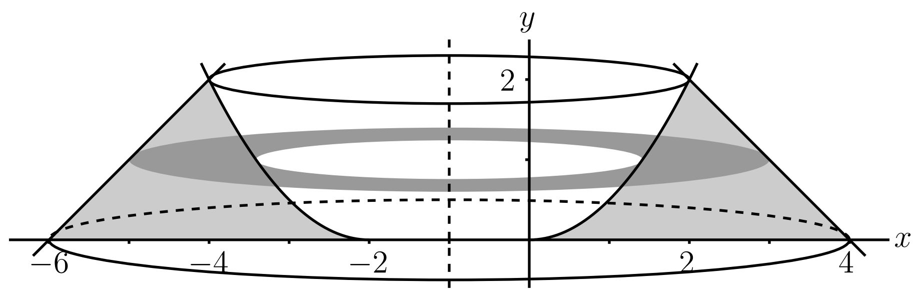


The shell with radius x has height $\dfrac{1}{x^2}$, so its volume is $2\pi \dfrac{x}{x^2} \Delta x$. The volume of the solid is given by

$$\int_1^2 \frac{2\pi x}{x^2}\, dx \quad \text{or} \quad 2\pi \int_1^2 \frac{dx}{x}$$

7. Let R be the region in the first quadrant bounded by the x-axis, the line $x + y = 4$, and the curve $2y = x^2$. Set up, but do not evaluate, an integral for the volume of the solid generated when R is revolved about the line $x = -1$.

Solution: Here is a picture of the region and the solid it generates.



To find the point in the first quadrant where $x+y = 4$ and $2y = x^2$ intersect, we solve both equations for y, and set the two y expressions equal to one another, getting

$$4 - x \quad = \quad \frac{x^2}{2}$$

We simplify this to get $x^2 + 2x - 8 = 0$, and factor to get

$$(x + 4)(x - 2) \quad = \quad 0$$

The intersection point in the first quadrant is $(2, 2)$.

The integral for the volume will be simpler if we use the washer method. The washer at height y has its center at $x = -1$ and its outer edge along the line $x = 4 - y$, so its outer radius is $(4 - y) - (-1)$.

The washer at height y has its center at $x = -1$ and its inner edge along the curve $x = \sqrt{2y}$, so its inner radius is $\sqrt{2y} - (-1)$.

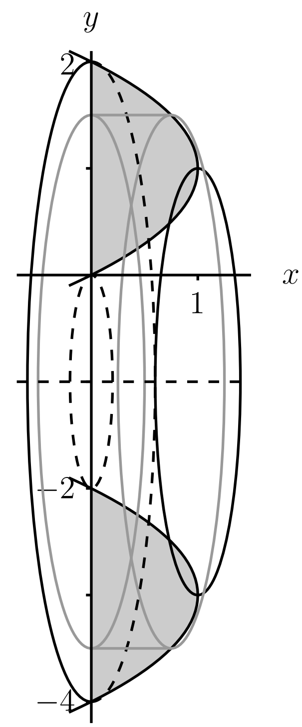
The volume of the washer at height y is

$$\pi(((4 - y) + 1)^2 - (\sqrt{2y} + 1)^2)\Delta y \quad = \quad \pi((5 - y)^2 - (\sqrt{2y} + 1)^2)\Delta y$$

so the volume of the solid is

$$\int_0^2 \pi((5 - y)^2 - (\sqrt{2y} + 1)^2)\, dy$$

8. Let R be the region bounded by the y-axis and the parabola $x = 2y - y^2$. Set up, but do not evaluate, an integral for the volume of the solid generated when R is revolved about the line $y = -1$.



Solution: A drawing of the region and the solid is at right. We will use the shell method. The curve

$$x = 2y - y^2$$

intersects the y-axis at the points $(0, 0)$ and $(0, 2)$, so we can identify the shells making up the object by y-coordinates from $y = 0$ to $y = 2$.
The shell at position y on the y-axis has radius

$$y - (-1) \quad = \quad y + 1$$

and height

$$2y - y^2$$

so its volume is

$$2\pi(y + 1)(2y - y^2)\,\Delta y$$

The volume of the entire solid is

$$\int_0^2 2\pi(y + 1)(2y - y^2)\,dy$$