Math 202(01) Solution
Practice Exam II 21 March 2004

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1. The amount of work required to stretch a certain spring 3 meters beyond its natural length is 15 Newton-meters. Find the spring constant for the spring. (Remember to specify the units.)

   Solution: Let k denote the spring constant. Then we know

$$15 \text{ N-m} \quad = \quad \int_0^3 kx\,dx$$

   with x in meters and k in Newtons per meter. Evaluating the integral (and suppressing the units for the moment), we get

$$\begin{aligned} 15 &= \left[\frac{kx^2}{2}\right]_0^3 \\ &= \frac{9k}{2} \end{aligned}$$

   so that $k = \dfrac{30}{9}$ Newtons per meter.

2. A chain six feet long and weighing 2 pounds per foot lies in a coil on the ground. How much work is required to lift one end of the chain to a height of six feet?

   Solution: When the upper end of the chain is at a height x, the force required to hold it up is x feet times 2 pounds per foot, or $2x$ pounds. The work required to pull the end of the chain through a distance Δx at height x is thus $2x\,\Delta x$ foot-pounds.

   The total amount of work W required to lift the end of the chain through 6 feet is given by

$$\begin{aligned} \int_0^6 2x\,dx &= \left[x^2\right]_0^6 \\ &= 36 \text{ foot-pounds.} \end{aligned}$$

3. Compute $\displaystyle\int x\sec^2 x\,dx$

   Solution: We use integration by parts. We have

$$\begin{aligned} u &= x & \qquad v &= \tan x \\ du &= dx & \qquad dv &= \sec^2 x\,dx \end{aligned}$$

Applying the integration by parts formula, we get

$$\begin{aligned} \int x \sec^2 x \, dx &= x \tan x - \int \tan x \, dx \\ &= x \tan x - \ln|\sec x| + C \end{aligned}$$

4. Evaluate $\int_0^{\frac{\pi}{2}} x \cos x \, dx$

   Solution: We use integration by parts. We have

$$\begin{aligned} u &= x & \qquad v &= \sin x \\ du &= dx & \qquad dv &= \cos x \, dx \end{aligned}$$

   We apply the integration by parts formula to get

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \cos x \, dx &= [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx \\ &= \left(\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right)\right) + [\cos x]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} + (0 - 1) \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

5. Compute $\int \sin^3 x \cos^3 x \, dx$

   Solution: We write

$$\begin{aligned} \int \sin^3 x \cos^3 x \, dx &= \int \sin^3 x \cos^2 x (\cos x) \, dx \\ &= \int \sin^3 x (1 - \sin^2 x) \cos x \, dx \end{aligned}$$

   and let $u = \sin x$, so that $du = \cos x \, dx$. We get

$$\begin{aligned} \int \sin^3 x (1 - \sin^2 x) \cos x \, dx &= \int u^3 (1 - u^2) \, du \\ &= \int u^3 - u^5 \, du \\ &= \frac{u^4}{4} - \frac{u^6}{6} + C \\ &= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C \end{aligned}$$

6. Compute $\int \sqrt{1+2x^2}\,dx$

Solution: We want a substitution that makes $2x^2 = \tan^2\theta$. The substitution is

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}\tan\theta \\ dx &= \frac{1}{\sqrt{2}}\sec^2\theta\,d\theta \end{aligned}$$

Making this substitution, we get

$$\begin{aligned} \int \sqrt{1+2x^2}\,dx &= \int \sqrt{1+\tan^2\theta}\cdot\frac{1}{\sqrt{2}}\sec^2\theta\,d\theta & (1) \\ &= \frac{1}{\sqrt{2}}\int \sec^3\theta\,d\theta & (2) \end{aligned}$$

To evaluate $\int \sec^3\theta\,d\theta$, we apply integration by parts with

$$\begin{aligned} u &= \sec\theta & \qquad v &= \tan\theta \\ du &= \sec\theta\tan\theta\,d\theta & \qquad dv &= \sec^2\theta\,d\theta \end{aligned}$$

We get

$$\begin{aligned} \int \sec^3\theta\,d\theta &= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta \\ &= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\,d\theta \\ &= \sec\theta\tan\theta + \int \sec\theta\,d\theta - \int \sec^3\theta\,d\theta \\ &= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta\,d\theta \end{aligned}$$

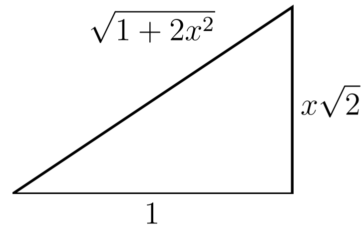
We add $\int \sec^3\theta\,d\theta$ to both sides and solve to get

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C$$

We now substitute this expression into line (2) to get

$$\frac{1}{2\sqrt{2}}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C$$

Finally, we need to undo the substitutions. Using the fact that $\tan\theta = x\sqrt{2}$, we find that $\sec\theta = \sqrt{1+2x^2}$.



The integral is

$$\begin{aligned} &\frac{1}{2\sqrt{2}}\left(x\sqrt{2}\sqrt{1+2x^2} + \ln|\sqrt{1+2x^2} + x\sqrt{2}|\right) + C \\ &= \frac{x\sqrt{1+2x^2}}{2} + \frac{1}{2\sqrt{2}}\ln(\sqrt{1+2x^2} + x\sqrt{2}) + C \end{aligned}$$

7. Compute $\displaystyle\int \frac{x^3}{x^2+2x+4}\,dx$.

Solution: To begin, we carry out the long division. We have

$$\begin{array}{r|l} & x - 2 \\ \hline x^2+2x+4 & x^3 \\ & x^3 + 2x^2 + 4x \\ \hline & \quad - 2x^2 - 4x \\ & \quad - 2x^2 - 4x - 8 \\ \hline & \qquad\qquad\qquad 8 \end{array}$$

We have

$$\int \frac{x^3}{x^2+2x+4}\,dx = \int x - 2 + \frac{8}{x^2+2x+4}\,dx \tag{3}$$

$$= \frac{x^2}{2} - 2x + 8\int \frac{dx}{x^2+2x+4} \tag{4}$$

We complete the square in the denominator in the remaining integral, getting

$$x^2 + 2x + 4 = (x+1)^2 + 3$$

To compute $\displaystyle\int \frac{dx}{(x+1)^2+3}$ we use the substitution $(x+1) = \sqrt{3}\tan\theta$, so that $dx = \sqrt{3}\sec^2\theta\,d\theta$. We get

$$\int \frac{dx}{(x+1)^2+3} = \int \frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta + 3}\,d\theta$$

$$
\begin{aligned}
&= \int \frac{\sqrt{3}\sec^2\theta}{3\sec^2\theta}\,d\theta \\
&= \frac{1}{\sqrt{3}}\int d\theta \\
&= \frac{1}{\sqrt{3}}\theta + C
\end{aligned}
$$

From $\tan\theta = \dfrac{x+1}{\sqrt{3}}$, we get

$$\theta \;=\; \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right),$$

so the integral evaluates to $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{3}}\right)$. Substituting this back into (4), we get

$$\int \frac{x^3}{x^2+2x+4}\,dx \;=\; \frac{x^2}{2} - 2x + \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

8. Compute $\displaystyle\int \frac{8+7x-x^2}{x(x+2)^2}\,dx$

   Solution: We use partial fractions. We write

$$\frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \;=\; \frac{8+7x-x^2}{x(x+2)^2}$$

Multiplying through to clear denominators, we get

$$
\begin{aligned}
8+7x-x^2 &= A(x+2)^2 + Bx(x+2) + Cx \\
&= Ax^2 + 4Ax + 4A + Bx^2 + 2Bx + Cx \\
&= (A+B)x^2 + (4A+2B+C)x + 4A
\end{aligned}
$$

This implies that $4A = 8$, $A + B = -1$, and $4A + 2B + C = 7$. From the first of these equations, we get $A = 2$. Then the second implies that $B = -3$. We substitute these values into the third equation to get $8 - 6 + C = 7$, so that $C = 5$. The integral is

$$\int \frac{2}{x} - \frac{3}{x+2} + \frac{5}{(x+2)^2}\,dx \;=\; 2\ln|x| - 3\ln|x+2| - \frac{5}{x+2} + C$$

9. Use the trapezoid rule with $n = 40$ subintervals to estimate $\int_0^2 \frac{1}{1+x^3}\,dx$. Give bounds on the error in your estimate.

Solution: The width of each subinterval is $(2-0)/40 = 1/20$, so the subintervals are

$$\left[0, \frac{1}{20}\right], \left[\frac{1}{20}, \frac{2}{20}\right], \left[\frac{2}{20}, \frac{3}{20}\right], \ldots, \left[\frac{39}{20}, 2\right]$$

We have

$$\begin{aligned} T_{40} &= \frac{1}{2\times 20}\left[f(0) + 2f\left(\frac{1}{20}\right) + 2f\left(\frac{2}{20}\right) + \cdots + 2f\left(\frac{39}{20}\right) + f(2)\right] \\ &= \frac{1}{40}\left(1+\frac{1}{9}\right) + \frac{2}{40}\left[f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + \cdots + f\left(\frac{39}{20}\right)\right] \end{aligned}$$

where $f(x) = \frac{1}{1+x^3}$.

The calculator say that this is approximately 1.08997082.

To find error bounds, we need to find K_2, and to find K_2, we need to find $f''(x)$. We have

$$\begin{aligned} f'(x) &= -\frac{3x^2}{(1+x^3)^2} \\ f''(x) &= -\frac{(1+x^3)^2(6x) - 2(3x^2)^2(1+x^3)}{(1+x^3)^4} \\ &= \frac{18x^4 - 6x(1+x^3)}{(1+x^3)^3} \\ &= \frac{12x^4 - 6x}{(1+x^3)^3} \end{aligned}$$

Graphing this on a calculator, we note that the furthest the curve gets from the x-axis is about 1.7375. If we take this as our K_2 and apply the error formula (with $n = 40$, $a = 0$, and $b = 2$), we get

$$\begin{aligned} E_T &\leq \frac{K_2(2)^3}{12(40)^2} \\ &\approx 0.00072396 \end{aligned}$$

So the actual value of the integral must lie somewhere between

$$1.08997082 - 0.00072396 \quad \text{and} \quad 1.08997082 + 0.00072396,$$

that is, between 1.08924686 and 1.09069478.

10. Suppose we use the midpoint rule to estimate $\int_1^2 \frac{1}{x}\,dx$. What is the smallest value of n that will guarantee an error of less than 10^{-4}?

Solution: Let $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$. The maximum value of $|f''(x)|$ is $|f''(1)| = 2$. The error bound, as a function of n, is thus

$$\begin{aligned} E &\leq \frac{2(2-1)^3}{24n^2} \\ &= \frac{1}{12n^2} \end{aligned}$$

For $E \leq 10^{-4}$, we need

$$\begin{aligned} \frac{1}{12n^2} &\leq 10^{-4} \\ \frac{10^4}{12} &\leq n^2 \\ \frac{10^2}{\sqrt{12}} &\leq n \end{aligned}$$

so we need $n \geq \frac{100}{\sqrt{12}} \approx 28.87$. We'd have to take $n = 29$ to guarantee the desired accuracy.