Math 202(01) Solution
Practice Exam III 19 April 2004

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1. Use Simpson's rule with $n = 8$ to estimate $\int_1^3 \frac{dx}{x}$. Give a bound on the error in your estimate.

   Solution: The subintervals are

   $$\left[\frac{4}{4}, \frac{5}{4}\right], \left[\frac{5}{4}, \frac{6}{4}\right], \left[\frac{6}{4}, \frac{7}{4}\right], \dots, \left[\frac{11}{4}, \frac{12}{4}\right]$$

   The sum for Simpson's rule is

   $$\frac{1}{3 \times 4}\left[f\left(\frac{4}{4}\right) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 4f\left(\frac{7}{4}\right) + \cdots + 4f\left(\frac{11}{4}\right) + f\left(\frac{12}{4}\right)\right]$$

   which becomes

   $$\frac{1}{12}\left[1 + 4\cdot\frac{4}{5} + 2\cdot\frac{4}{6} + 4\cdot\frac{4}{7} + 2\cdot\frac{4}{8} + 4\cdot\frac{4}{9} + 2\cdot\frac{4}{10} + 4\cdot\frac{4}{11} + \frac{4}{12}\right]$$

   The calculator says that this is approximately 1.09872535.

   To esimate the error, we need the fourth derivative of the function $f(x) = \frac{1}{x}$. We get

   $$f'(x) = -\frac{1}{x^2}; \quad f''(x) = \frac{2}{x^3}; \quad f'''(x) = -\frac{6}{x^4}; \quad f^{(4)}(x) = \frac{24}{x^5}$$

   The maximum value of $|f^{(4)}(x)|$ on $[1, 3]$ occurs at $x = 1$, and the value is 24. The error bound is thus

   $$\begin{aligned} E_S &\leq \frac{24(3-1)^5}{180(8)^4} \\ &\approx 0.00104167 \end{aligned}$$

2. Find the length of the curve $y = \frac{e^x + e^{-x}}{2}$ for $0 \leq x \leq 1$.

   Solution: We find that $y' = \frac{e^x - e^{-x}}{2}$, so that

   $$1 + (y')^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$
\begin{aligned}
&= \frac{e^{2x} + 2 + e^{-2x}}{4} \\
&= \frac{(e^x + e^{-x})^2}{4} \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2
\end{aligned}
$$

The length of the curve is

$$
\begin{aligned}
\int_0^1 \sqrt{1 + (y')^2}\, dx &= \int_0^1 \frac{e^x + e^{-x}}{2}\, dx \\
&= \frac{1}{2}\left[e^x - e^{-x}\right]_0^1 \\
&= \frac{1}{2}\left[(e - e^{-1}) - (1 - 1)\right] \\
&= \frac{e + e^{-1}}{2}
\end{aligned}
$$

3. Set up, but do not evaluate, an integral for the area of the surface generated when the part of the curve $y = \sin(\pi x)$ between $x = 0$ and $x = 1$ is revolved about the x-axis.

   Solution: We have $y' = \pi \cos(\pi x)$, so the area is given by

$$
\int_0^1 2\pi \sin(\pi x)\sqrt{1 + \pi^2 \cos^2(\pi x)}\, dx
$$

4. Compute $\displaystyle\int_{-\infty}^0 \frac{x+1}{(x-1)^2}\, dx$

   Solution: We rewrite the integral as

$$
\lim_{t \to -\infty} \int_t^0 \frac{x+1}{(x-1)^2}\, dx.
$$

   To evaluate the indefinite integral $\displaystyle\int \frac{x+1}{(x-1)^2}\, dx$ we try the substitution $u = x - 1$, $du = dx$. We have

$$
\begin{aligned}
\int \frac{x+1}{(x-1)^2}\, dx &= \int \frac{u+2}{u^2}\, du \\
&= \int \frac{1}{u} + \frac{2}{u^2}\, du
\end{aligned}
$$

$$\begin{aligned} &= \ln|u| - \frac{2}{u} + C \\ &= \ln|x-1| - \frac{2}{x-1} + C \end{aligned}$$

We now evaluate the improper integral, getting

$$\begin{aligned} \lim_{t\to-\infty}\left[\ln|x-1| - \frac{2}{x-1}\right]_t^0 &= \lim_{t\to-\infty}\left(\ln 1 - \frac{2}{-1} - \ln|t-1| + \frac{2}{t-1}\right) \\ &= 2 - \lim_{t\to-\infty}\ln|t-1| + \lim_{t\to-\infty}\frac{2}{t-1} \end{aligned}$$

The middle term goes to ∞, so the integral is divergent.

5. Compute $\displaystyle\int_1^{10} \frac{1}{(x-2)^{\frac{2}{3}}}\,dx$

Solution: The denominator is zero at $x = 2$, so this is an improper integral. We write it as

$$\int_1^{10} \frac{1}{(x-2)^{\frac{2}{3}}}\,dx = \lim_{t\to2^-}\int_1^t \frac{1}{(x-2)^{\frac{2}{3}}}\,dx + \lim_{t\to2^+}\int_t^{10} \frac{1}{(x-2)^{\frac{2}{3}}}\,dx$$

We first compute the indefinite integral. We get

$$\int (x-2)^{-\frac{2}{3}}\,dx = 3(x-2)^{\frac{1}{3}} + C$$

The first limit above becomes

$$\begin{aligned} \lim_{t\to2^-}\left[3(x-2)^{\frac{1}{3}}\right]_1^t &= \lim_{t\to2^-}\left(3(t-2)^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}}\right) \\ &= 0 - 3(-1) \\ &= 3 \end{aligned}$$

The second limit is

$$\begin{aligned} \lim_{t\to2^+}\left[3(x-2)^{\frac{1}{3}}\right]_t^{10} &= \lim_{t\to2^+}\left(3\cdot 8^{\frac{1}{3}} - 3(t-2)^{\frac{1}{3}}\right) \\ &= 3\cdot 2 - 0 \\ &= 6 \end{aligned}$$

The value of the original integral is $3 + 6 = 9$.

6. Waiting times at the TechSupport hotline are distributed exponentially. Records show that 25% of callers to TechSupport have to wait 5 minutes or less. What fraction of callers have to wait 10 minutes or more?

Solution: The probability density function has the form $\frac{1}{\mu}e^{-\frac{t}{\mu}}$. The fact that 25% of callers have to wait 5 minutes or less tells us that

$$\begin{aligned} \frac{1}{4} &= \int_0^5 \frac{1}{\mu}e^{-\frac{t}{\mu}}\,dt \\ &= \left[-e^{-\frac{t}{\mu}}\right]_0^5 \\ &= 1 - e^{-\frac{5}{\mu}} \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{4} &= 1 - e^{-\frac{5}{\mu}} \\ e^{-\frac{5}{\mu}} &= \frac{3}{4} \\ -\frac{5}{\mu} &= \ln\left(\frac{3}{4}\right) \\ \mu &= -\frac{5}{\ln(3/4)} \end{aligned}$$

The fraction of callers who have to wait 10 minutes or more is given by

$$\begin{aligned} \int_{10}^{\infty} \frac{1}{\mu}e^{-\frac{t}{\mu}}\,dt &= \lim_{w\to\infty}\left[-e^{-\frac{t}{\mu}}\right]_{10}^{w} \\ &= e^{-\frac{10}{\mu}} \end{aligned}$$

We can substitute in our value of μ from above to get

$$\begin{aligned} e^{-\frac{10}{\mu}} &= e^{2\ln(3/4)} \\ &= e^{\ln((3/4)^2)} \\ &= \frac{9}{16} \end{aligned}$$

The fraction of callers who have to wait longer than 10 minutes is $\frac{9}{16}$.

7. Find the exact value of $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{4}{7}\right)^{n}$

Solution: The series begins

$$\frac{4}{7}-\left(\frac{4}{7}\right)^{2}+\left(\frac{4}{7}\right)^{3}-\left(\frac{4}{7}\right)^{4}+\cdots$$

We write this as

$$\frac{4}{7}\left[1+\left(-\frac{4}{7}\right)+\left(-\frac{4}{7}\right)^{2}+\left(-\frac{4}{7}\right)^{2}+\cdots\right]$$

and use the formula for the sum of a geometric series to conclude that

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{4}{7}\right)^{n} &= \frac{4}{7}\cdot\frac{1}{1+\frac{4}{7}}\\ &= \frac{4}{11}\end{aligned}$$

8. Determine whether each of the following series is convergent or divergent. Give reasons for your conclusions.

   (a) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^{3}+n}}$

   Solution: We use the Basic Comparison Test. For $n \geq 1$, we have $\sqrt{n^{3}+n} > \sqrt{n^{3}}$, so that

$$\begin{aligned}\frac{1}{\sqrt{n^{3}+n}} &< \frac{1}{\sqrt{n^{3}}}\\ &= \frac{1}{n^{\frac{3}{2}}}\end{aligned}$$

   We know that $\sum\frac{1}{n^{\frac{3}{2}}}$ converges, because it is a p-series with $p > 1$. Therefore, the given series converges by Basic Comparison with $\sum\frac{1}{n^{\frac{3}{2}}}$.

   (b) $\sum_{n=1}^{\infty}\cos\left(\frac{1}{n^{2}}\right)$

Solution: Since $\lim_{n\to\infty} \frac{1}{n^2} = 0$ and the cosine function is continuous at 0, we have

$$\begin{aligned} \lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) &= \cos(0) \\ &= 1 \\ &\neq 0 \end{aligned}$$

so the series diverges by the Divergence Test.

(c) $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$

Solution: We apply the Integral Test. We have

$$\begin{aligned} \int_0^{\infty} \frac{x}{x^2+1}\,dx &= \lim_{t\to\infty} \int_0^t \frac{x}{x^2+1}\,dx \\ &= \lim_{t\to\infty} \frac{1}{2}\left[\ln(x^2+1)\right]_0^t \\ &= \lim_{t\to\infty} \ln(t^2+1) \end{aligned}$$

which diverges. The original series diverges by the Integral Test.

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+2}}{n^2}$

Solution: We apply the Basic Comparison Test. Since $\sqrt{n^2+2} > \sqrt{n^2}$ for all n, we know that

$$\begin{aligned} \frac{\sqrt{n^2+2}}{n^2} &> \frac{\sqrt{n^2}}{n^2} \\ &= \frac{1}{n} \end{aligned}$$

for all n. We know that the harmonic series diverges, so the given series also diverges, by Basic Comparison with the harmonic series.