1. Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Give reasons.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n}{n3^{2n+1}}$

   Solution: We test for absolute convergence using the ratio test. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)3^{2n+3}} \frac{n3^{2n+1}}{4^n} \\ &= \lim_{n\to\infty} \frac{4n}{9(n+1)} \\ &= \frac{4}{9}\end{aligned}$$

   Since $\frac{4}{9} < 1$, the series converges absolutely.

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n} + \frac{e^{-n}}{n}\right)$

   Solution: We first apply the alternating series test. We observe that e^{-n} decreases as n increases, and so $\frac{e^{-n}}{n}$ decreases as n increases. Of course $\frac{1}{n}$ decreases as n increases also. Thus the absolute values of the given series form a decreasing sequence.

   Moreover, $\displaystyle\lim_{n\to\infty} \frac{1}{n} + \frac{e^{-n}}{n} = 0 + 0$. Thus the given series converges by the alternating series test.

   Does it converge absolutely? We note that

$$\frac{1}{n} + \frac{e^{-n}}{n} > \frac{1}{n}$$

   for all $n \geq 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Thus by basic comparison with the harmonic series, the given series does not converge absolutely. We conclude that the given series is conditionally convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n+5}}{n(n+1)}$

Solution: The alternating series test would be something of a hassle here (do the absolute values of the terms really decrease?) so we'll start by testing for absolute convergence.

The term $\dfrac{\sqrt{n+5}}{n(n+1)}$ should be asymptotic to $\dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{\frac{3}{2}}}$ as $n \to \infty$, so we'll try limit comparison with $\dfrac{1}{n^{\frac{3}{2}}}$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n+5}}{n(n+1)} \frac{n^{\frac{3}{2}}}{1} &= \lim_{n\to\infty} \sqrt{\frac{(n+5)n^3}{n^2(n+1)^2}} \\
&= \sqrt{\lim_{n\to\infty} \frac{n^4+5n^3}{n^4+2n^3+n^2}} \\
&= \sqrt{1} \\
&= 1
\end{aligned}$$

Thus by limit comparison, the absolute version of the given series does the same thing as the series $\sum \dfrac{1}{n^{\frac{3}{2}}}$. Since this is a p-series with $p > 1$, we know it converges. The given series is absolutely convergent.

2. Find a power series for the function $f(x) = \dfrac{x}{(x+2)^2}$. Give the interval of convergence.

Solution: Let's start with $\dfrac{1}{2+x}$. We multiply top and bottom by $\dfrac{1}{2}$ to get

$$\frac{1}{2+x} = \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\frac{1}{1-(-\frac{x}{2})}$$

Using the formula for the sum of a geometric series, we have

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{2}\frac{1}{1-(-\frac{x}{2})} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}
\end{aligned}$$

for $\left|\frac{x}{2}\right| < 1$, which is to say for $|x| < 2$. We take derivatives of both sides to get

$$\begin{aligned}
-\frac{1}{(2+x)^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n n x^{n-1}}{2^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n n x^{n-1}}{2^{n+1}}
\end{aligned}$$

so that

$$\frac{1}{(x+2)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{n-1}}{2^{n+1}}$$

for $|x| < 2$. Next we multiply this series by x to get

$$\frac{x}{(x+2)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{n}}{2^{n+1}} \quad \text{for } |x| < 2$$

3. Find the interval of convergence for each power series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^{2n}}{n^2+1}$

   Solution: To begin, we apply the ratio test. We get

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{|x+2|^{2n+2}}{(n+1)^2+1} \frac{n^2+1}{|x+2|^{2n}} \\
&= \lim_{n\to\infty} |x+2|^2 \frac{n^2+1}{(n+1)^2+1} \\
&= |x+2|^2
\end{aligned}$$

   We solve $|x+2|^2 < 1$, getting $-3 < x < -1$.
   Next we check the endpoints. At $x = -3$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

   which converges by basic comparison with the p-series $\sum \frac{1}{n^2}$.

At $x = -1$, we have

$$\sum_{n=0}^{\infty} \frac{1^{2n}}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

which, again, converges.

The interval of convergence for the original series is $[-3, -1]$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$.

Solution: We begin with the ratio test. We get

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)3^{n+1}} \frac{n3^n}{|x-1|^n} \\
&= \lim_{n\to\infty} \frac{|x-1|}{3} \frac{n}{n+1} \\
&= \frac{|x-1|}{3}
\end{aligned}$$

We solve $\dfrac{|x-1|}{3} < 1$, getting $-2 < x < 4$.

Next we check the endpoints. At $x = -2$, we have

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is the alternating harmonic series, and it converges by the alternating series test.

At $x = 4$, we have

$$\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the harmonic series, which we know to be divergent.

The interval of convergence for the original series is $[-2, 4)$.

4. Use a power series to estimate the value of $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1+2x^2}\,dx$ with an error of less than 10^{-4}.

Solution: First we find a power series for the integrand. We have

$$\begin{aligned}\frac{1}{1+2x^2} &= \frac{1}{1-(-2x^2)}\\ &= \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n}\end{aligned}$$

for $|2x^2| < 1$. The interval of convergence is thus $|x| < \dfrac{1}{\sqrt{2}}$, and since both limits of the integral above fall in this interval, we can integrate the power series and use it to estimate the value of the integral.

We get

$$\int \frac{1}{1+2x^2}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^{2n+1}}{2n+1} + C$$

so that

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{1+2x^2}\,dx &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(2n+1)2^{2n+1}} - 0\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)2^{n+1}}\\ &= \frac{1}{2} - \frac{1}{3\times 4} + \frac{1}{5\times 8} - \frac{1}{7\times 16} + \frac{1}{9\times 32} - \cdots\end{aligned}$$

To get the desired accuracy, we need to find the first term in the series whose absolute value is less than 10^{-4}. By trial and error, we find that the first such term is $\dfrac{1}{19\times 1024}$, which is the $n = 9$ term.

Our estimate should therefore be the sum of the terms for $n = 0$ through $n = 8$. We get

$$\frac{1}{2} - \frac{1}{3\times 4} + \frac{1}{5\times 8} - \cdots + \frac{1}{17\times 512} \approx 0.435245$$

5. Find the fourth-order Taylor polynomial for $f(x) = (x+1)^{\frac{2}{3}}$ about $a = 0$.

   Use it to estimate the value of $(1.1)^{\frac{2}{3}}$. Give an upper bound for the error in your estimate.

Solution: We have

$$\begin{aligned}
f(x) &= (x+1)^{\frac{2}{3}} \\
f'(x) &= \frac{2}{3}(x+1)^{-\frac{1}{3}} \\
f''(x) &= -\frac{2}{3^2}(x+1)^{-\frac{4}{3}} \\
f'''(x) &= \frac{2\times 4}{3^3}(x+1)^{-\frac{7}{3}} \\
f^{(4)}(x) &= -\frac{2\times 4\times 7}{3^4}(x+1)^{-\frac{10}{3}}
\end{aligned}$$

so that

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= \frac{2}{3} \\
f''(0) &= -\frac{2}{9} \\
f'''(0) &= \frac{8}{27} \\
f^{(4)}(0) &= -\frac{56}{81}
\end{aligned}$$

The Taylor polynomial $T_4(x)$ is given by

$$\begin{aligned}
T_4(x) &= 1+\frac{2}{3}x-\frac{2}{9\times 2!}x^2+\frac{8}{27\times 3!}x^3-\frac{56}{81\times 4!}x^4 \\
&= 1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3-\frac{7}{243}x^4
\end{aligned}$$

The value of $(1.1)^{\frac{2}{3}}$ is approximated by $T_4(0.1)$, which is

$$1+\frac{2}{30}-\frac{1}{900}+\frac{4}{81000}-\frac{7}{2430000} \approx 1.06560206$$

Since the series alternates, the error in this approximation is less than the first term we omitted. That term would have been

$$\begin{aligned}
\frac{2\times 4\times 7\times 10}{3^5}\frac{1}{10^5\times 5!} &= \frac{560}{243\times 120\times 10^5} \\
&\approx 1.92\times 10^{-7}
\end{aligned}$$

6. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n3^n}{10^{n+1}}$

Solution: Let s denote the sum. Then we have

$$\begin{aligned} s &= \frac{3}{100} \sum_{n=0}^{\infty} \frac{n3^{n-1}}{10^{n-1}} \\ &= \frac{3}{100} \sum_{n=0}^{\infty} nx^{n-1} \end{aligned}$$

with $x = \frac{3}{10}$.

Since we know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$, and we know that we can take derivatives on both sides, we get

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Our original sum is therefore

$$\begin{aligned} \frac{3}{100} \frac{1}{(1-\frac{3}{10})^2} &= \frac{3}{100} \frac{100}{49} \\ &= \frac{3}{49} \end{aligned}$$