Trigonometry Worksheet

1. Pythagorean Theorem refresher

If a right triangle has legs of lengths a and b and hypotenuse of length c, then $a^2 + b^2 = c^2$.

1.1 Fill in the missing length on each of the following right triangles. (NOTE: Don't use decimals. For example, write $1/2$ and not 0.5; write $\sqrt{3}$ and not $1.732\ldots$; write $\sqrt{(3)}/2$ and not $0.866\ldots$.

[3 triangles with hypotenuse 1, one with leg labelled $\sqrt{3}/2$, one with leg labelled $\sqrt{2}/2$, and one with leg labelled x.]

1.2 (i) The figure below shows a point with coordinates (x, y) on the *unit circle* (i.e., the circle of radius 1 centered at the origin) and in the first quadrant. Label the sides of the indicated right triangle.

[figure here]

1.2 (ii) The figure below shows a point with coordinates $(-x, y)$ on the unit circle and in the second quadrant. Draw a right triangle with the line segment from the origin to the point as its hypotenuse and label the lengths of the sides.

[figure here]

1.2 (iii) and (iv) Draw appropriate figures in the third and fourth quadrants and label the lengths of the sides. [As in (i) and (ii), the values of the positive numbers x and y aren't specified, but draw these two figures so that they appear to have the same values of x and y as those in (i) and (ii).]

1.3 What is the equation of the unit circle?

2. Radian measure

An angle of 1 *radian* is defined to be the engle at the center of a unit circle which cuts off an arc of length 1 measured counter-clockwise along the circle.

[picture here]

An angle of measure θ (in radians) determines a point P_θ on the unit circle as follows:

- If $\theta \geq 0$, P_θ is the endpoint of an arc of length θ beginning at $(1, 0)$ and going counter-clockwise along the circle. (So the arc from $(1, 0)$ to P_θ determines a central angle of measure θ.)
- If $\theta < 0$, P_θ is the endpoint of an arc of length $|\theta|$ beginning at $(1, 0)$ and going clockwise along the circle. (So the arc from $(1, 0)$ to P_θ again determines a central angle of measure θ.)

2.1 On the unit circle on the next page, label the points P_θ for the following values of θ. (Some points will end up with multiple labels.)

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π	$9\pi/4$	$5\pi/2$	3π	$7\pi/2$	4π

2.2 On the same unit circle as for 2.1, label the points P_θ for the following values of θ. (Some points will end up with multiple labels.)

θ	-4π	$-7\pi/2$	-3π	$-5\pi/2$	$-9\pi/4$	-2π	$-7\pi/4$	$-3\pi/2$	$-5\pi/4$	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$

2.3 Using the right triangles in section 1 as needed, label each point on the unit circle on the next page with its coordinates. For example, the coordinates of $P_{3\pi/2}$ are $(0, -1)$.

3. Defining sine and cosine

The functions sine and cosine are defined using a unit circle and radian measure for angles by the following procedure. Given the angle θ, determine a point P_θ on the unit circle as in section 2. By definition,

$$\begin{aligned} \cos\theta &= x\text{-coordinate of } P_\theta \\ \sin\theta &= y\text{-coordinate of } P_\theta \end{aligned}$$

3.1 Using your results in 2.1–2.3, fill in the following table of values of the sine and cosine for positive angles.

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π	$9\pi/4$	$5\pi/2$	3π	$7\pi/2$	4π
$\cos\theta$														
$\sin\theta$														

3.2 Using your results in 2.1–2.3, fill in the following table of values of the sine and cosine for negative angles.

θ	-4π	$-7\pi/2$	-3π	$-5\pi/2$	$-9\pi/4$	-2π	$-7\pi/4$	$-3\pi/2$	$-5\pi/4$	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$
$\cos\theta$													
$\sin\theta$													

3.3 Using your tables of values in 3.1 and 3.2, draw the graph of $y = \cos x$ for $-4\pi \le x \le 4\pi$.

3.4 Using your tables of values in 3.1 and 3.2, draw the graph of $y = \sin x$ for $-4\pi \le x \le 4\pi$.

3.5 Does the unit circle definition of the sine and cosine agree with the right triangle definition for nonzero acute angles?

Right triangle definition. Draw a right triangle containing an acute angle of measure θ. Let H be the length of the hypotenuse, A be the length of the leg adjacent to the angle of measure θ and O be the length of the leg opposite the angle of measure θ. Then $\cos\theta$ is defined to equal A/H and $\sin\theta$ is defined to equal O/H.

Note that a property of *similar* triangles is needed to establish that these definitions make sense. If you draw a second right triangle containing an acute angle of measure θ, the first and second triangles are similar because the measures of all three angles of the second triangle agree with the measures of all three angles of the first. It follows that the legs of the second triangle have lengths rA and rO and the hypotenuse has length rH for some fixed positive number r. So, for example, if you calculate $\cos\theta$ using the second triangle, you get $rA/rH = A/H$, which agrees with your result in the first triangle.

4. Some trigonometric identities

Answer the following questions by referring to your picture in section 2.

4.1 What is the relationship between $\cos\theta$ and $\cos(-\theta)$?

4.2 What is the relationship between $\sin\theta$ and $\sin(-\theta)$?

4.3 What is the relationship between $\cos\theta$ and $\cos(\theta + 2\pi)$?

4.4 What is the relationship between $\sin\theta$ and $\sin(\theta + 2\pi)$?

4.5 What is the relationship between $\cos\theta$ and $\sin(\theta + \pi/2)$?

4.6 What is the value of $(\cos\theta)^2 + (\sin\theta)^2$?

4.7 What is the largest value that $\cos\theta$ can have? The smallest? What about $\sin\theta$?

5. Properties of the graphs of $y = \cos x$ and $y = \sin x$.

5.1 How do the relationships in 4.1 and 4.3 show in your graph of $y = \cos x$? Is your graph consistent with the maximum and minimum values in 4.7?

5.2 How do the the relationships in 4.2 and 4.4 show in your graph of $y = \sin x$? Is your graph consistent with the maximum and minimum values in 4.7?

5.3 How does the relationship in 4.5 show in your graphs of $y = \sin x$ and $y = \cos x$?