Claim 1 *For any real number $r \neq 1$ and every integer $n \geq 0$,*

$$\sum_{k=0}^{n} r^k \;=\; \frac{1-r^{n+1}}{1-r} \tag{1}$$

Proof: For each non-negative integer n, let

$$T_n \;=\; 1 + r + r^2 + \cdots + r^n$$

Then we have

$$\begin{aligned}(1-r)T_n &= (1-r)(1+r+r^2+\cdots+r^n)\\ &= \begin{array}{ccccccccccc} 1 & + & r & + & r^2 & + & \cdots & + & r^n & & \\ & - & r & - & r^2 & - & \cdots & - & r^n & - & r^{n+1}\end{array}\\ &= 1 - r^{n+1}\end{aligned}$$

so that

$$(1-r)T_n \;=\; 1 - r^{n+1} \tag{2}$$

Now since $r \neq 1$, we can divide both sides of (2) by $1-r$ to get

$$T_n \;=\; \frac{1-r^{n+1}}{1-r}$$

and since $T_n = \sum_{k=0}^{n} r^k$, we have proved (1). ∎

Alternate Proof: We proceed by induction on n.

For the base case, take $n = 0$. On the left side of (1), we have $\sum_{k=0}^{0} r^k$, which is just r^0, or 1 (we are assuming that $0^0 = 1$ for this problem). On the right side of (1), we have $\frac{1-r}{1-r}$, which, for $r \neq 1$, is also equal to 1.

Thus, (1) holds in the base case.

For the inductive step, we assume that (1) holds for some value of $n \geq 0$, and show that

$$\sum_{k=0}^{n+1} r^k = \frac{1 - r^{(n+1)+1}}{1 - r} \tag{3}$$

We have

$$\begin{aligned} \sum_{k=0}^{n+1} r^k &= \left(\sum_{k=0}^{n} r^k \right) + r^{n+1} && (4) \\ &= \left(\frac{1 - r^{n+1}}{1 - r} \right) + r^{n+1} && (5) \end{aligned}$$

by the inductive hypothesis.

Now the right side of (5) is just

$$\begin{aligned} \frac{1 - r^{n+1}}{1 - r} + r^{n+1} &= \frac{1 - r^{n+1} + (1 - r) r^{n+1}}{1 - r} && (6) \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{(n+1)+1}}{1 - r} && (7) \\ &= \frac{1 - r^{(n+1)+1}}{1 - r} && (8) \end{aligned}$$

Putting lines (4) through (8) together, we get

$$\sum_{k=0}^{n+1} r^k = \frac{1 - r^{(n+1)+1}}{1 - r}$$

which is just line (3). Thus, by the principle of mathematical induction, claim (1) holds for all integers $n \geq 0$. ■