A Putnam problem on integer sums

Gregory Quenell

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1 Introduction

Problem A-1 of the 2003 William Lowell Putnam Mathematical Competition ([3]) reads as follows

> Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, $2 + 2$, $1 + 1 + 2$, $1 + 1 + 1 + 1$.

In this brief note, we will experiment with sums of the form given in Problem A-1, guess how many such sums there are for each positive integer n, and eventually prove that our guess is correct.

2 Notation

For a positive integer n, let $S(n)$ denote the number of ways that n can be written as a sum $n = a_1 + a_2 + \cdots + a_k$ with

$$a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1 \tag{1}$$

where the a_i are all positive integers. Note that condition (1) implies that every a_i is either equal to a_1 or to $a_1 + 1$, so $S(n)$ is just the number of ways that n can be written as a sum

$$n = (a_1 + a_1 + \cdots + a_1) + ((a_1 + 1) + (a_1 + 1) + \cdots + (a_1 + 1)) \qquad (2)$$

where a_1 is some positive integer. We make the following definition.

Definition A sum will be called *neighboring* if it has the form of the sum in (2).

3 Experimental data

For small values of n, we can determine $S(n)$ simply by listing all the ways we can write n as a sum of the form (2).

- For $n = 1$, we can write only the trivial sum $1 = 1$.
- For $n = 2$, there are two neighboring sums: $2 = 2$ and $2 = 1 + 1$.
- For $n = 3$, we have

$$\begin{aligned} 3 &= 3 \\ &= 2 + 1 \\ &= 1 + 1 + 1 \end{aligned}$$

  for a total of three neighboring sums.
- For $n = 4$, the four neighboring sums are given in the statement of the problem in section 1
- For $n = 5$, we have

$$\begin{aligned} 5 &= 5 \\ &= 3 + 2 \\ &= 2 + 2 + 1 \\ &= 2 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 \end{aligned}$$

for a total of five neighboring sums. (No neighboring sum for 5 could include a 4, because it would then have to include either a 3 or another 4).

We summarize these data in the table below.

n	1	2	3	4	5
$S(n)$	1	2	3	4	5

4 Two conjectures

The obvious conjecture from the data we collected in section 3 is

Conjecture 1 *For each positive integer n, the number of ways to write n as a neighboring sum is n.*

Looking more closely at the data, we notice another pattern. For each $n \in \{1, 2, 3, 4, 5\}$, each of the neighboring sums we wrote down has a different number of terms. Since the number of terms in a sum for n cannot exceed n, this leads to another conjecture.

Conjecture 2 *Given a positive integer n, for each positive integer $k \leq n$, there is exactly one way to write n as neighboring sum with k terms.*

Why should this be so? Consider $n = 5$, say, and suppose we want to write 5 as a neighboring sum with three terms. Then each of the three terms has to be approximately 5/3. Since 5/3 is between 1 and 2, we guess that each of the three terms in the neighboring sum will be either 1 or 2. We get $5 = 2 + 2 + 1$. If we want to write 5 and a two-term neighboring sum, we know each of the terms will be either 2 or 3, since 5/2 lies between 2 and 3. Clearly $5 = 2 + 3$ is the sum we want.

To turn this idea into a proof of Conjecture 2, we will need a standard theorem about integer division.

5 The Division Algorithm

Despite the word "algorithm" in its name, the Division Algorithm is actually a theorem. It says (see [2, page 5], for example)

Theorem 3 (The Division Algorithm) *Given a positive integer a and an integer b, there exist unique integers q and r with $0 \leq r < a$ such that $b = qa + r$.*

Using Theorem 3, we can prove Conjecture 2, which we state a little more precisely as

Theorem 4 *Given a positive integer n and a positive integer $k \leq n$, there is a unique neighboring sum for n with k terms.*

Proof Given n and k as in the statement of the theorem, use the Division Algorithm to write $n = kq + r$ with $0 \leq r < k$. Since $n \geq k$ and $r < k$, it follows that $kq = n - r$ is positive, and so $q \geq 1$.

We rewrite the equation $n = kq + r$ as

$$\begin{aligned} n &= \underbrace{q + q + \cdots + q}_{k \text{ terms}} + \underbrace{1 + 1 + \cdots + 1}_{r \text{ terms}} && (3) \\ &= \underbrace{q + q + \cdots + q}_{k-r \text{ terms}} + \underbrace{(q+1) + (q+1) + \cdots + (q+1)}_{r \text{ terms}} && (4) \end{aligned}$$

Since q is a positive integer, line (4) expresses n as a neighboring sum with k terms.

To see that this expression is unique, suppose n and k are as in the statement of the theorem, and that

$$n = \underbrace{q + q + \cdots + q}_{k-r \text{ terms}} + \underbrace{(q+1) + (q+1) + \cdots + (q+1)}_{r \text{ terms}} \qquad (5)$$

and

$$n = \underbrace{q' + q' + \cdots + q'}_{k-r' \text{ terms}} + \underbrace{(q'+1) + (q'+1) + \cdots + (q'+1)}_{r' \text{ terms}} \qquad (6)$$

are two neighboring sums for n. Then from (5), we have

$$n = (k - r)q + r(q + 1) = kq + r \tag{7}$$

and from (6), we have

$$n = (k - r')q' + r'(q' + 1) = kq' + r' \tag{8}$$

Lines (7) and (8) imply that $n = kq + r = kq' + r'$ where $0 \le r < k$ and $0 \le r' < k$. But the numbers q and r in the Division Algorithm are uniquely determined, so we must have $q = q'$ and $r = r'$. Thus expressions (5) and (6) are identical. ∎

6 Conclusion

From Theorem 4 we get the following corollary, which is the same as Conjecture 1.

Corollary 5 *For each positive integer n, there are exactly n ways to write n as a neighboring sum.*

Proof By Theorem 4, for each $k \in \{1, 2, \ldots, n\}$, there is exactly one way to write n as a neighboring sum with k terms. Since there are n elements in the set $\{1, 2, \ldots, n\}$, there are a total of n ways to write n as a neighboring sum. ∎

We remark that neighboring sums are closely related to maximally even sets (see [1] for a definition), and that Theorem 4 can be used to construct an algorithm ([4]) that generates maximally even sets.

References

[1] Clough, John and Gerald Myerson, "Musical scales and the generalized circle of fifths," *American Mathematical Monthly* 93(9), 1986, pp. 695-701.

[2] Niven, Ivan, Herbert Zuckerman, and Hugh Montgomery, *An Introduction to the Theory of Numbers*, fifth edition, John Wiley & Sons, Inc., 1991.

[3] The Sixty-Fourth William Lowell Putnam Mathematical Competition, administered by the Mathematical Association of America, Washington DC.

[4] Quenell, Gregory, "Recursive smooth sums and maximally even sets," in preparation.