Read Chapter 2 sections 5 and 6!! Carefully!

1. High levels of cholesterol in the blood are known to be a risk factor for heart disease. Cholesterol is manufactured by the body for use in the construction of cell walls and is absorbed from foods containing cholesterol. The following is a very simple model of blood cholesterol levels. Let $C(t)$ be the amount of cholesterol in the blood (in milligrams per deciliter) of a particular person at time $t$ days. Then

$$\frac{dC}{dt} = k_1 (C_0 - C) + k_2 E$$

where $C_0$ is the person’s “natural” cholesterol level, $k_1$ is a production parameter (set by experiment and meant to quantify how fast the body is producing cholesterol), $E$ is the amount of cholesterol eaten per day, and $k_2$ is an “absorption” parameter (set by experiment and meant to quantify how fast the body is absorbing cholesterol from the food).

a. Suppose $C_0 = 200$, $k_1 = 0.1$, $k_2 = 0.1$, $E = 400$, and $C(0) = 150$. What will the person’s cholesterol be after two days on this diet?

b. With the initial conditions from part a, what will the person’s cholesterol be after five days on this diet?

c. With the initial conditions from part a, what will the person’s cholesterol be after a very long time on this diet?

d. Suppose that, after a very long time on the high cholesterol diet described above, the person goes on a very low cholesterol diet, so $E$ changes to $E = 100$. (Take your answer to part c above as your initial cholesterol level for this second diet.) What will the person’s cholesterol be after one day on this second diet? After five days on this second diet? After a very long time on this second diet?

e. Suppose the person stays on the high cholesterol diet but takes drugs that block some of the uptake of cholesterol from food, so $k_2$ changes to $k_2 = 0.075$. With the cholesterol level from part c as your initial cholesterol level, what will the person’s cholesterol level be after one day taking the drug? After five days on the drug? After a very long time on the drug?

2. For the differential equations below find all their equilibria and determine their stability (stable vs unstable). Explain your reasoning. You may use a direction field plot to check yourself but give reasons that do not cite the plot.

a. 

$$\frac{dp}{dt} = p^2 - 2p - 8$$
b. \[ \frac{dp}{dt} = p^2 - \frac{16}{p^2} \]

3. **The Gompertz model**: In 1825 Gompertz proposed the following population model:

\[ \frac{dp}{dt} = r p \ln\left( \frac{K}{p} \right) \]

with \( p(0) = p_0 \) where \( p(t) \) is the population size at time \( t \), \( r > 0 \) is the intrinsic growth rate, \( K > 0 \) is the carrying capacity for the population, and \( p_0 > 0 \) is the initial population. Note: does this model have the three properties that our book says we should look for in a population model (top of page 76): Is \( r \ln\left( \frac{K}{p} \right) \) close to \( r \) for small \( p \)? Does \( r \ln\left( \frac{K}{p} \right) \) decrease as \( p \) get larger? Is \( r \ln\left( \frac{K}{p} \right) \) less than 0 for \( p \) sufficiently large?

a. Show that \( p = K \) is an asymptotically stable equilibrium.

b. Solve the initial value problem above. Hint: use the substitution \( u = \ln\left( \frac{K}{p} \right) \) and derive a differential equation for \( u(t) \). If you do this right you will get a very familiar linear ODE. Solve this ODE for \( u \) and from that obtain your solution \( p(t) \).

c. Using your solution from part b, show that as \( t \to \infty \), \( p(t) \to K \), as long as \( p_0 > 0 \).

d. In class (and in the book) we solved and analyzed solutions of the logistic population growth model:

\[ \frac{dp}{dt} = r p \left( 1 - \frac{p}{K} \right) \]

where \( p(0) = p_0 \). Now we can compare the Gompertz model with the logistic model. Take \( K = 1000 \) and \( r = 1 \) in both models. Choose a couple of values for \( p_0 < 1000 \) and a couple of values for \( p_0 > 1000 \) and plot the solutions of the Gompertz and the logistic DE with these values. Comment on the differences and the similarities in the behavior of the solutions. What biological insight do you gain from the comparisons?

4. **Logistic model with constant yield harvesting**: Consider a population that grows logistically, but is subjected to harvesting (or removal) in which \( H \) individuals are harvested per unit time, \( H > 0 \) (\( H \) is a constant). The Initial value problem (IVP) describing this situation is

\[ \frac{dp}{dt} = r p \left( 1 - \frac{p}{K} \right) - H \]

where \( p(0) = p_0 \). Here \( r \) is the intrinsic growth rate and \( K \) is the carrying capacity.
a. Find the two equilibria for this model?

b. Actually, the two equilibria that you found above only exist (i.e., are real numbers) under certain conditions on the model parameters $H$, $K$, and $r$. What are these conditions? Can you make sense of the conditions from a biological point of view?

c. Show that when no equilibria exist, $\frac{dp}{dt} < 0$ for all $t$.

What happens in this case is that the population hits 0 (i.e., becomes extinct) in finite time. Thus, constant yield harvesting can have catastrophic consequences on the population if the conditions found in part b for the existence of positive equilibria are violated. One example to which this might apply is to the over-fishing of some species of fish. The point is one has to be careful not to catch the fish at too great a rate.

d. Now examine the case in which positive equilibria do exist. Examine the stability of each of the equilibria.

e. Use the phaseportrait command from Maple to plot some solutions for this model when $K = 1000, r = 1, p_0 = 100$ and several values for $H$ that satisfy the conditions you found in part b. Also plot solutions for the logistic equation for these parameter values (but with $H = 0$ so there is no harvesting). Compare an comment on the results. Do your plots of the solutions agree with your results from part d about stability?

Extra credit f. Note that you have done this whole problem without solving the harvesting ODE. You can actually solve it analytically with partial fractions. For extra credit solve this IVP in the case when the conditions of part b are met.