Examples of polynomial knots

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Abstract

In this paper, we define and give examples of polynomial knots. In particular, we write down specific polynomial equations with rational coefficients for seven different knots, ranging from the figure eight knot to a knot with ten crossings. A nice property that all of these knots share is that they have planar projections that are symmetric about the y-axis. This research was conducted at Mt. Holyoke college, during the summer of 2004. It was supported by the NSF, through grant DMS-0353700.

1 Introduction

Definition 1. A *polynomial knot* is a smooth embedding of $\mathbb{R}$ in $\mathbb{R}^3$ defined by

$$t \mapsto (f(t), g(t), h(t))$$

where $f(t)$, $g(t)$ and $h(t)$ are polynomials over the field of real numbers.

Polynomial knots were first looked at by Anant R. Shastri, who subsequently wrote a paper summarizing his results entitled *Polynomial Representations of Knots* [1]. In his paper, he proved that every knot type has a polynomial representation. He demonstrates this by constructing polynomial equations for specific knot types, including the trefoil knot (the knot with three crossings) and the figure eight knot (the knot with four crossings). In particular, he defines the trefoil by a map $\varphi\colon \mathbb{R} \to \mathbb{R}^3$, given by

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the parametric equations [1]:

$$\begin{aligned} f(t) &= t^3 - 3t \\ g(t) &= t^4 - 4t^2 \\ h(t) &= t^5 - 10t \end{aligned}$$

and he defines the figure eight knot by a map $\psi\colon \mathbb{R} \to \mathbb{R}^3$, given by the parametric equations [1]:

$$\begin{aligned} f(t) &= t^3 - 3t \\ g(t) &= t(t^2 - 1)(t^2 - 4) \\ h(t) &= t^7 - 42t \end{aligned}$$

Shastri's exploration of the properties of polynomial knots, as well as his simple mappings for the trefoil and figure eight knots, has opened the door for further research on the subject. Our research group was particularly interested in finding additional mappings for polynomial knots with nice properties, i.e. mappings with binomial equations (like Shastri's for a trefoil knot), as well as mappings that gave planar projections of knots that are symmetric about the y-axis. What follows are some examples of polynomial knots which include some of these nice properties.

2 An example of a polynomial knot with binomial equations

Looking at Shastri's equations, we see that his mapping for the trefoil knot includes polynomial equations $f(t)$, $g(t)$ and $h(t)$ that are each binomial, yet his mapping for the figure eight knot includes two binomial equations $f(t)$ and $h(t)$, and one non-binomial equation $g(t)$. This led our research group to believe that we would always have a mapping with at least one non-binomial equation for any knot other than the trefoil knot, since Shastri's equations appeared to be as simple as they could get. However, we discovered that this is not the case.

Proposition 1. It is possible to find a mapping that includes only binomial equations for a polynomial knot which is different from the trefoil knot, in particular, for the figure eight knot.

Proof: To show this, we must first find two binomial equations (preferably of low degree) that give us a projection of the figure eight knot. So we

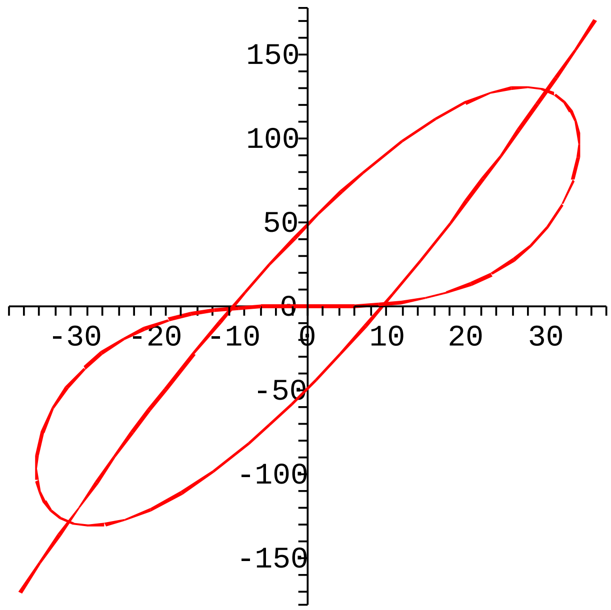

Figure 1: XY projection of the figure eight knot

consider the equations:

$$\begin{aligned} x(t) &= t^5 - 28t \\ y(t) &= t^7 - 32t^3 \end{aligned}$$

and look at their projection on the XY plane. As one can see from figure 1, this is clearly a projection of the figure eight knot.

Next, we must check that these equations have the right under(over) crossing data to give us the figure eight knot. To do this, we must find the t-values that give the four double points $t_i \neq t_j$ in $\mathbb{R}$ such that $x(t_i) = x(t_j)$, $y(t_i) = y(t_j)$ and $t_i < t_{i+1}$. We can solve for the double points algebraically, by first looking at the equations:

$$\begin{aligned} x(t_i) - x(t_j) &= 0 \\ y(t_i) - y(t_j) &= 0 \end{aligned}$$

and then diving through by $t_i - t_j$, since we assumed that $t_i \neq t_j$. The double points are then the solutions to the equations:

$$\begin{aligned}\frac{x(t_i) - x(t_j)}{t_i - t_j} &= 0 \\ \frac{y(t_i) - y(t_j)}{t_i - t_j} &= 0\end{aligned}$$

However, we decide to calculate the t-values that yield the double points by using a computer:

$$\begin{aligned}(t_1, t_6) &= (-2.51564, 1.84397) \\ (t_2, t_5) &= (-2.38028, .348779) \\ (t_3, t_8) &= (-1.84397, 2.51564) \\ (t_4, t_7) &= (-.348779, 2.38028)\end{aligned}$$

Next, we must use the fact that the figure eight knot is an alternating knot.

Definition 2. An *alternating knot* is a knot with a projection that has crossings that alternate between over and under as one travels around the knot in a fixed direction [2].

So we must find a polynomial equation $z(t)$ that makes this knot with four crossings alternate, and thus gives us the figure eight knot.

To do this, we choose our first crossing, at (t_1, t_6) to be an under crossing, so that for a polynomial $z(t)$:

$$z(t_1) < z(t_6) \tag{1}$$

Since the knot must alternate at the crossings, it must follow that:

$$\begin{aligned}z(t_2) &> z(t_5) &(2)\\ z(t_3) &< z(t_8) &(3)\\ z(t_4) &> z(t_7) &(4)\end{aligned}$$

Now, in order to try to find a $z(t)$ that satisfies the four inequalities above, we first choose values in between all of the t values, which are preferably rational and as simple as possible. Then we allow $z(t)$ to have zeroes at those values. This gives us the equation:

$$z(t) = t\left(t + \frac{5}{2}\right)\left(t - \frac{5}{2}\right)(t+2)(t-2)(t+1)(t-1)$$

which simplifies to:

$$z(t) \quad = \quad t^7 - \frac{45}{4}t^5 + \frac{141}{4}t^3 - 25t \tag{5}$$

It is easy to check that equation (5) satisfies the inequalities(1), (2), (3), and (4) defined above, so we may proceed.

Since $y(t)$, like $z(t)$, is a polynomial of degree seven, we can subtract $y(t)$ from $z(t)$ to make $z(t)$ a polynomial of lower degree.

We can do this because the map:

$$(x, y, z) \to (x, y, z - y)$$

is an invertible linear map of $\mathbb{R}^3 \to \mathbb{R}^3$. Applying this map to $\mathbb{R}^3$ takes the polynomial knot $(x(t), y(t), z(t))$ to the equivalent polynomial knot $(x(t), y(t), z(t) - y(t))$. As a result, we get a new polynomial equation for $z(t)$, which has a smaller degree than before:

$$z(t) - y(t) = t^7 - \frac{45}{4}t^5 + \frac{141}{4}t^3 - 25t - (t^7 - 32t^3) = t^5 - \frac{329}{45}t^3 + \frac{20}{9}t$$

So now we have a $z(t)$ equation of degree five, and notice that since $x(t)$ is of degree five as well, we can similarly apply another map:

$$(x, y, z - y) \to (x, y, z - y - x)$$

which is also an invertible linear map of $\mathbb{R}^3 \to \mathbb{R}^3$. The end result is that:

$$z(t) = t^3 - \frac{1360}{329}t$$

We can not apply any more maps to reduce $z(t)$ to a lower degree, so we know that z(t) is a polynomial of degree three. However, we notice that $4 < \frac{1360}{329} < 5$, and since we would prefer that $z(t)$ have integral coefficients, we try:

$$z(t) = t^3 - 5t \tag{6}$$

It is easy to check that equation (6), like equation (5), satisfies the four inequalities (1), (2), (3), and (4) defined above, and thus provides the necessary under(over) crossing data.

Therefore, the mapping $\phi\colon \mathbb{R} \to \mathbb{R}^3$ defined by the parametric equations:

$$\begin{aligned} x(t) &= t^5 - 28t \\ y(t) &= t^7 - 32t^3 \\ z(t) &= t^3 - 5t \end{aligned}$$

gives us a figure eight knot, as desired. Furthermore, we notice that all of these equations are binomial, and thus we have shown that we can find a mapping with exclusively binomial equations that gives us a knot different from the trefoil. □

3 Other examples of polynomial knots

Along with being defined by a map with nice binomial equations, we notice that the figure eight knot we have just described has another nice property: it has a planar projection which is symmetric about the y-axis. What follows are some more polynomial knots which share this property.

3.1 Another mapping which defines the figure eight knot

First we consider the equations

$$\begin{aligned} x(t) &= t(t^2-7)(t^2-10) \\ y(t) &= t^4-13t^2 \end{aligned}$$

and look at their projection on the XY plane. We can clearly see from figure 2 that this is a projection of the figure eight knot.

Next, we use a computer to calculate the t-values that yield the double points:

$$\begin{aligned} (t_1,t_4) &= (-3.52379,-.763502) \\ (t_2,t_7) &= (-\sqrt{10},\sqrt{10}) \\ (t_3,t_6) &= (-\sqrt{7},\sqrt{7}) \\ (t_5,t_8) &= (.763502,3.52379) \end{aligned}$$

and can easily check that $z(t)$ satisfies the inequalities necessary to alternate the projection at the four crossings:

$$\begin{aligned} z(t_1) &< z(t_4) \\ z(t_2) &> z(t_7) \\ z(t_3) &< z(t_6) \\ z(t_5) &< z(t_8) \end{aligned}$$

and thus provides us with the right under(over) crossing data.

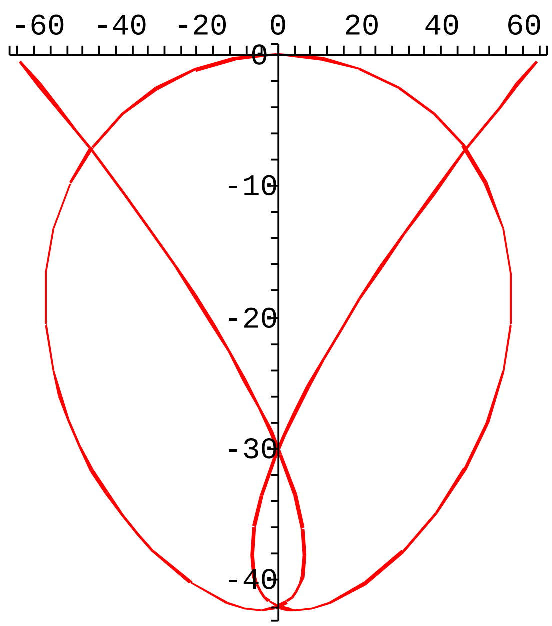

Figure 2: XY projection of the figure eight knot

So the mapping $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ defined by the parametric equations:

$$
\begin{aligned}
x(t) &= t(t^2-7)(t^2-10) \\
y(t) &= t^4-13t^2 \\
z(t) &= t(t^2-4)(t^2-9)\left(t^2-\frac{49}{4}\right)
\end{aligned}
$$

provides us with another figure eight knot, as desired.

3.2 A mapping for a knot with five crossings

The mapping $\vartheta\colon \mathbb{R} \to \mathbb{R}^3$ defined by the parametric equations:

$$
\begin{aligned}
x(t) &= t^5-36t^3+260t \\
y(t) &= t^4-24t^2 \\
z(t) &= t^7-31t^5+168t^3+560t
\end{aligned}
$$

provides us with another knot, but this time it is not the figure eight. It is a knot with five crossings, which can easily be seen from the projection of the $x(t)$ and $y(t)$ equations in figure 3.

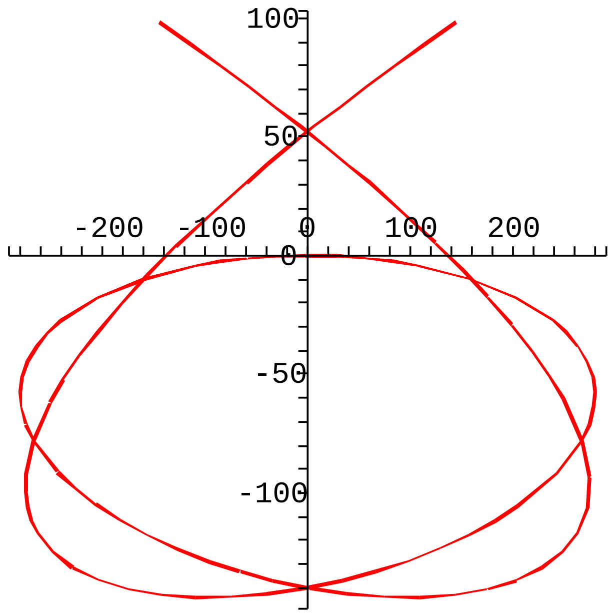

Figure 3: XY projection of a knot with five crossings

To verify that we actually have a knot with five crossings, we must check that $z(t)$ satisfies the right under(over) crossing data.

So we solve for the double points:

$$
\begin{aligned}
(t_1, t_6) &= (-4.85523, .653228) \\
(t_2, t_7) &= (-4.48639, 1.96783) \\
(t_3, t_8) &= (-\sqrt{10}, \sqrt{10}) \\
(t_4, t_9) &= (-1.96783, 4.48639) \\
(t_5, t_{10}) &= (-.653228, 4.85523)
\end{aligned}
$$

and check that $z(t)$ satisfies these inequalities:

$$\begin{aligned} z(t_1) &< z(t_6) \\ z(t_2) &> z(t_7) \\ z(t_3) &< z(t_8) \\ z(t_4) &> z(t_9) \\ z(t_5) &< z(t_{10}) \end{aligned}$$

thus giving us a knot with five crossings.

3.3 A mapping for a knot with six crossings

The mapping $\zeta\colon \mathbb{R} \to \mathbb{R}^3$ defined by the parametric equations:

$$\begin{aligned} x(t) &= t(t^2-4)(t^2-11) \\ y(t) &= t^4-12t^2 \\ z(t) &= t(t^2-1)(t^2-9)\left(t^2-\frac{49}{16}\right)\left(t^2-\frac{169}{16}\right)\left(t^2-\frac{100}{9}\right) \end{aligned}$$

provides us with yet another knot, this time a knot with six crossings. The $x(t)$ and $y(t)$ equations clearly give us a projection of a knot with six crossings, as we can see in in figure 5. To verify that we actually have a knot with six crossings, we must check that $z(t)$ satisfies the right under(over) crossing data.

So we solve for the double points:

$$\begin{aligned} (t_1,t_6) &= (-3.42836,-.496319) \\ (t_2,t_{11}) &= (-\sqrt{11},\sqrt{11}) \\ (t_3,t_8) &= (-3.11892,1.50743) \\ (t_4,t_9) &= (-2,2) \\ (t_5,t_{10}) &= (-1.50743,3.11892) \\ (t_7,t_{12}) &= (.496319,3.42836) \end{aligned}$$

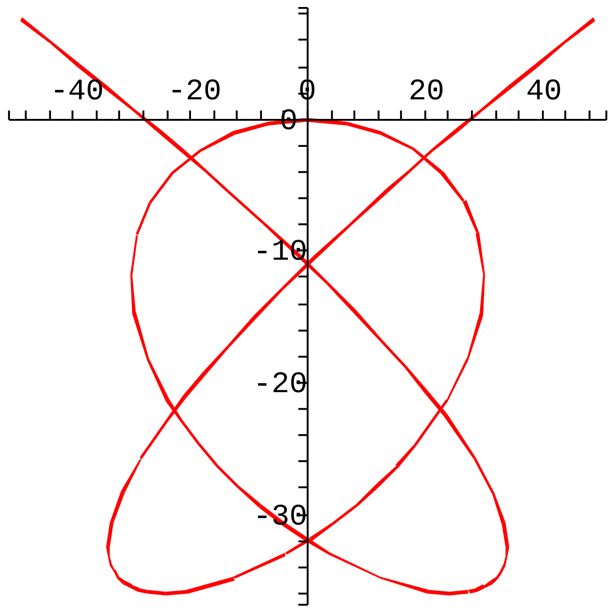

Figure 4: XY projection of a knot with six crossings

and check that $z(t)$ satisfies these inequalities:

$$\begin{aligned} z(t_1) &< z(t_6) \\ z(t_2) &> z(t_{11}) \\ z(t_3) &< z(t_8) \\ z(t_4) &> z(t_9) \\ z(t_5) &< z(t_{10}) \\ z(t_7) &< z(t_{12}) \end{aligned}$$

thus giving us a knot with six crossings.

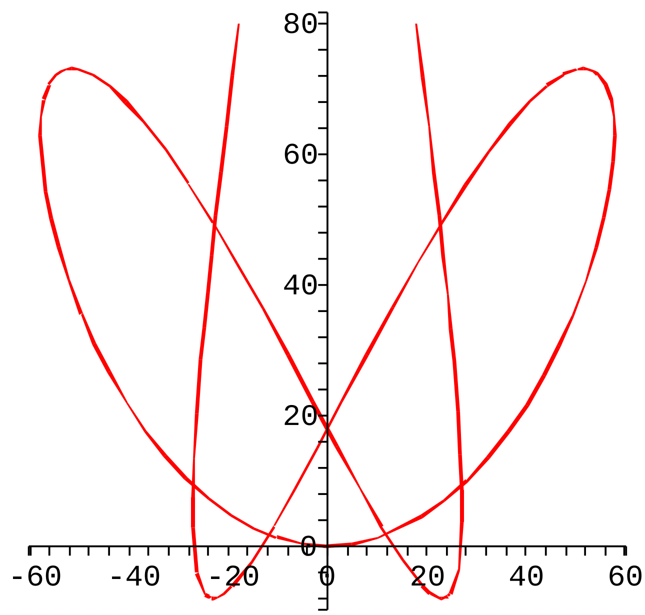

Figure 5: XY projection of a knot with seven crossings

3.4 A mapping for a knot with seven crossings

Similarly, we check that the mapping $\lambda\colon \mathbb{R} \to \mathbb{R}^3$, given by the parametric equations:

$$\begin{aligned} x(t) &= t(t^2-6)(t^2-12) \\ y(t) &= t^2(t^2-7)(t^2-9) \\ z(t) &= t(t^2-1)(t^2-9)\left(t^2-\frac{1}{16}\right)\left(t^2-\frac{81}{16}\right)\left(t^2-\frac{25}{4}\right)\left(t^2-\frac{256}{25}\right) \end{aligned}$$

gives us a knot, but with even more crossings. Seven in fact! The $x(t)$ and $y(t)$ equations clearly give us a projection of a knot with seven crossings, as we can see in in figure 5.

To verify that we actually have a knot with seven crossings, we must check that $z(t)$ satisfies the right under(over) crossing data.

So we solve for the double points:

$$\begin{aligned}
(t_1, t_{10}) &= (-3.22413, 2.13734) \\
(t_2, t_9) &= (-3.06739, .391134) \\
(t_3, t_8) &= (-2.6234, .165645) \\
(t_4, t_{11}) &= (-\sqrt{6}, \sqrt{6}) \\
(t_5, t_{14}) &= (-2.13734, 3.22413) \\
(t_6, t_{13}) &= (-.391134, 3.06739) \\
(t_7, t_{12}) &= (-.165645, 2.6234)
\end{aligned}$$

and check that $z(t)$ satisfies these inequalities:

$$\begin{aligned}
z(t_1) &< z(t_{10}) \\
z(t_2) &> z(t_9) \\
z(t_3) &< z(t_8) \\
z(t_4) &> z(t_{11}) \\
z(t_5) &< z(t_{14}) \\
z(t_6) &> z(t_{13}) \\
z(t_7) &< z(t_{12})
\end{aligned}$$

thus giving us a knot with seven crossings.

3.5 A mapping for a knot with eight crossings

Now, we check that the mapping $\varrho \colon \mathbb{R} \to \mathbb{R}^3$, given by the parametric equations:

$$\begin{aligned}
x(t) &= t^5(t^2 - 8)(t^2 - 10) \\
y(t) &= t^4 - 11t^2 \\
z(t) &= t(t^2 - 1)(t^2 - 4)(t^2 - 9)\left(t^2 - \frac{25}{16}\right)\left(t^2 - \frac{961}{100}\right)\left(t^2 - \frac{3969}{400}\right)\left(t^2 - \frac{361}{36}\right)
\end{aligned}$$

gives us a knot. The $x(t)$ and $y(t)$ equations clearly give us a projection of a knot with eight crossings, as we can see in in figure 6. To verify that we actually have a knot with eight crossings, we must check that $z(t)$ satisfies

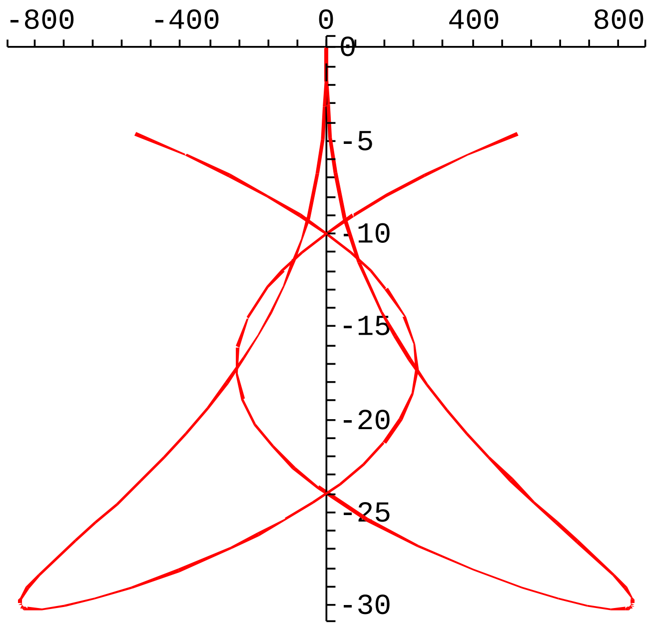

Figure 6: XY projection of a knot with eight crossings

the right under(over) crossing data. So we solve for the double points:

$$
\begin{aligned}
(t_1, t_8) &= (-3.17462, -.960092) \\
(t_2, t_15) &= (-\sqrt{10}, \sqrt{10}) \\
(t_3, t_{10}) &= (-3.13776, 1.07447) \\
(t_4, t_{11}) &= (-3.01574, 1.38032) \\
(t_5, t_{12}) &= (-\sqrt{8}, \sqrt{8}) \\
(t_6, t_{13}) &= (-1.38032, 3.01574) \\
(t_7, t_{14}) &= (-1.07447, 3.13776) \\
(t_9, t_{16}) &= (.960092, 3.17462)
\end{aligned}
$$

and check that $z(t)$ satisfies these inequalities:

$$
\begin{array}{rcl}
z(t_1) & < & z(t_8) \\
z(t_2) & > & z(t_{15}) \\
z(t_3) & < & z(t_{10}) \\
z(t_4) & > & z(t_{11}) \\
z(t_5) & < & z(t_{12}) \\
z(t_6) & > & z(t_{13}) \\
z(t_7) & < & z(t_{14}) \\
z(t_9) & > & z(t_{16})
\end{array}
$$

thus giving us a knot with eight crossings.

3.6 A mapping for a knot with ten crossings

Finally, we check that the mapping $\sigma\colon \mathbb{R} \to \mathbb{R}^3$, given by the parametric equations:

$$
\begin{array}{rcl}
x(t) & = & t(t^2-4)(t^2-9) \\
y(t) & = & t^2(t^2-7)(t^2-8) \\
z(t) & = & t(t^2-1)\left(t^2-\frac{1}{16}\right)\left(t^2-\frac{9}{4}\right)\left(t^2-\frac{441}{100}\right)\left(t^2-\frac{81}{16}\right) \\
 & & \left(t^2-\frac{25}{4}\right)\left(t^2-\frac{841}{100}\right)\left(t^2-\frac{3721}{400}\right)\left(t^2-\frac{961}{100}\right)
\end{array}
$$

gives a knot also, but this time a knot with unbelievably ten crossings! The $x(t)$ and $y(t)$ equations clearly give us a projection of a knot with ten crossings, as we can see in in figure 7.

To verify that we actually have a knot with eight crossings, we must check that $z(t)$ satisfies the right under(over) crossing data.

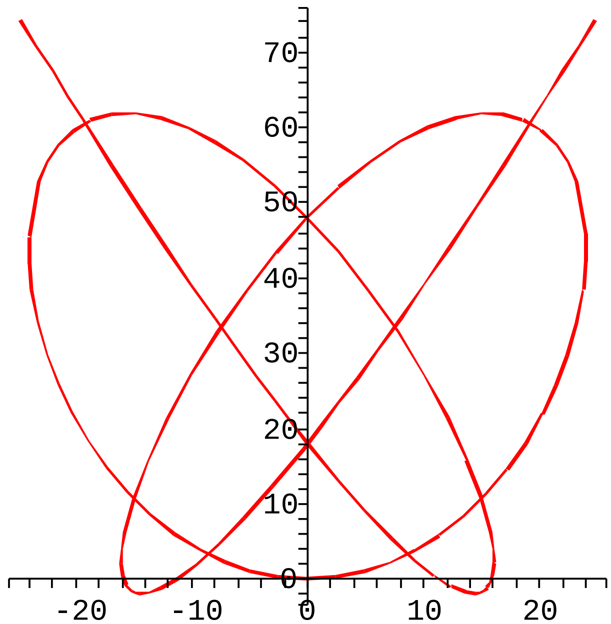

Figure 7: XY projection of a knot with ten crossings

So we solve for the double points:

$$
\begin{aligned}
(t_1, t_8) &= (-3.16383, -1.44056) \\
(t_2, t_{15}) &= (-3.07372, 2.19467) \\
(t_3, t_{18}) &= (-3, 3) \\
(t_4, t_{11}) &= (-2.88181, .244835) \\
(t_5, t_{12}) &= (-2.46673, .448512) \\
(t_6, t_{19}) &= (-2.19467, 3.07372) \\
(t_7, t_{14}) &= (-2, 2) \\
(t_9, t_{16}) &= (-448512, 2.46673) \\
(t_{10}, t_{17}) &= (-.244835, 2.88181) \\
(t_{13}, t_{20}) &= (1.44056, 3.16383)
\end{aligned}
$$

and check that $z(t)$ satisfies these inequalities:

$$\begin{array}{rcl} z(t_1) & < & z(t_8) \\ z(t_2) & > & z(t_{15}) \\ z(t_3) & < & z(t_{18}) \\ z(t_4) & > & z(t_{11}) \\ z(t_5) & < & z(t_{12}) \\ z(t_6) & > & z(t_{19}) \\ z(t_7) & < & z(t_{14}) \\ z(t_9) & < & z(t_{16}) \\ z(t_{10}) & > & z(t_{17}) \\ z(t_{13}) & < & z(t_{20}) \end{array}$$

thus giving us a knot with ten crossings.

References

[1] A. Shastri, *Polynomial Representations of Knots*. Tôhoku Math J. 44 (1992) 11-17.

[2] C. Adams, The Knot Book. W.H Freeman and Company (1994)